MATHEMATICS LEARNING AREA

YEAR 12 MATHEMATICS METHODS UNIT 3

Assessment type: Response

TASK 3- TEST 2

CALCULATOR- ASSUMED

:

Student Name: **ANSWER KEY**

TIME ALLOWED FOR THIS PAPER

Suggested:
Reading and Working time for Cal Assumed paper: 30 minutes in class under test conditions

MATERIAL REQUIRED / RECOMMENDED FOR THIS PAPER

TO BE PROVIDED BY THE SUPERVISOR
Question/answer booklet

TO BE PROVIDED BY THE CANDIDATE
Standard Items: pens, pencils, pencil sharpener, highlighter, eraser, ruler, drawing templates, Calculator

IMPORTANT NOTE TO CANDIDATES

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be attempted	Suggested working time (minutes)	Marks available
Calculator Assumed	4	4	30	**33**
			Marks available:	**/33**
			Task Weighting	7% for the pair of units

Instructions to candidates

Question 1 **(10 marks)**

A particle's velocity at t seconds, is $v(t) = 8\cos 2t$, and is travelling for 4π seconds.

a) Find the distance travelled by the particle for 4π seconds. (1 mark)

Solution

$Distance = \int_0^{4\pi} |8\cos 2t|\, dt$ (by using CAS)

$\therefore d = 64\, m$ (1 mark)

If the particle goes through the origin intially O.

b) Prove that $a(t) = -k^2 x(t)$. (3 marks)

Solution

$\int v(t)dt = x(t)$

$x(t) = 4\sin 2t + c \rightarrow x(0) = 0$

$\therefore x(t) = 4\sin 2t$ (1 mark)

$v'(t) = a(t) \rightarrow a(t) = -16\sin 2t$ (1 mark)

$a(t) = -(2)^2 4\sin 2t$

$\therefore a(t) = -(2)^2 x(t)$

Hence, the acceleration function is in the form of $a(t) = -k^2 x(t)$. (1 mark)

c) Find the exact speed of the particle at $t = \frac{11\pi}{12}$ (1 mark)

Solution

$\left|v\left(\frac{11\pi}{12}\right)\right| \rightarrow \left|8\cos 2\left(\frac{11\pi}{12}\right)\right| = 4\sqrt{3}$

$\therefore \left|v\left(\frac{11\pi}{12}\right)\right| = 4\sqrt{3}\, ms^{-1}$ (1 mark)

d) Find the change of displacement when the speed of the particle is $4\ ms^{-1}$ for the first $\frac{\pi}{2}$ seconds (5 marks)

Solution

$|v(t)| = 4$ (1 mark)

$\rightarrow v(t) = 4 \quad and \quad v(t) = -4$ (1 mark)

$8\cos 2t = 4\ |0 \leq t \leq \frac{\pi}{2}$

$\therefore t = \frac{\pi}{6}$ (1 mark)

$8\cos 2t = -4\ |0 \leq t \leq \frac{\pi}{2}$

$\therefore t = \frac{\pi}{3}$ (1 mark)

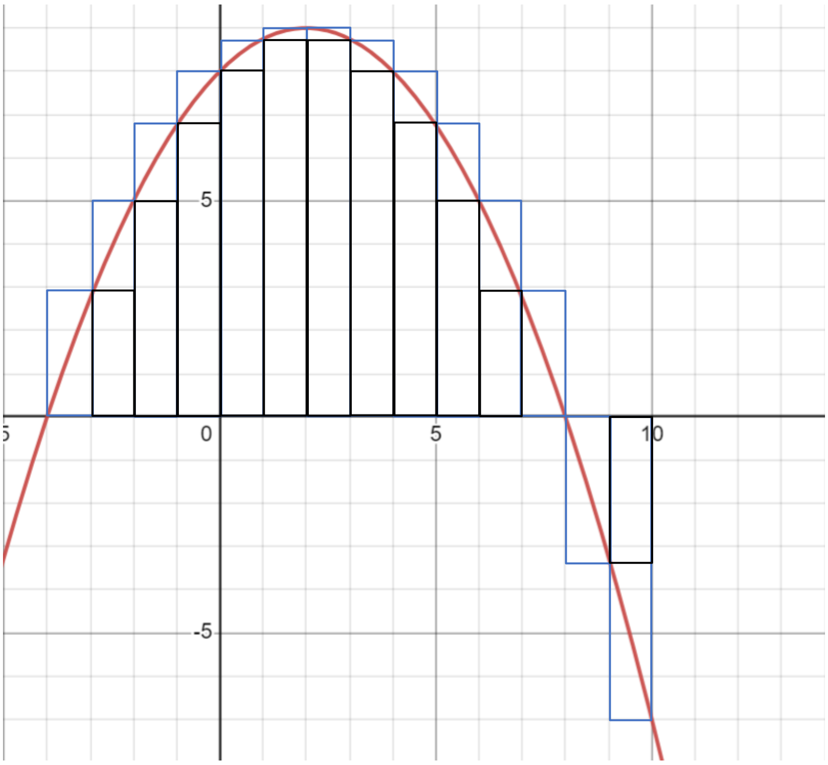
Change of displacement

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} v(t)dt \rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 8\cos 2t\ dt = 0$ (1 mark)

$\therefore$ Hence, the change of displacement is $0m$.

Question 2 **(6 marks)**

A function $x = f(y)$ is shown below.



Solution

Drawn circumscribed and inscribed rectangles (doesn't have to be colour coded) (1 mark)

Let area of inscribed rectangles have a width of 1 unit, and be represented as $A_{inscribed}$.

Let area of circumscribed rectangles have a width of 1 unit, and be represented as $A_{circumscribed}$.

To find area of inscribed and circumscribed, we must find the $f(y)$

$$f(y) = a(y-b)^2 + c$$
$$\rightarrow f(y) = a(y-2)^2 + 9$$

Sub in any point (e.g, use $(0,8)$)

It is given that $p < \int_{-4}^{10} |f(y)|\, dy < q$. By finding p and q, interpret what this statement ($p < \int_{-4}^{10} |f(y)|\, dy < q$) means by further on finding $\int_{-4}^{10} |f(y)|\, dy$.

Solution

$$8 = a(-2)^2 + 9$$
$$\therefore a = -\frac{1}{4}$$

$$f(y) = -\frac{1}{4}(y-2)^2 + 9$$ (1 mark)

$$A_{inscribed} = (1)\left[\left(\left(f(-3) + f(-2) + f(1) + f(0) + f(1)\right) \times 2\right) + |f(9)|\right]$$

$$\therefore A_{inscribed} = 65.75 units^2$$ (1 mark)

$$A_{circumscribed} = (1)\left[\left(\left(f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2)\right) \times 2\right) + |f(9)| + |f(10)|\right]$$

$$\therefore A_{circumscribed} = 90.75 units^2$$ (1 mark)

$$\int_{-4}^{10} \left| -\frac{1}{4}(y-2)^2 + 9 \right| dy = 78.\overline{66}$$ (1 mark)

Hence, $p = 65.75 units^2$ and $q = 90.75 units^2$

Furthermore, the actual area, $\int_{-4}^{10} |f(y)|\, dy$, lies within the range of $65.75 < \int_{-4}^{10} |f(y)|\, dy < 90.75$ as the $A_{inscribed}$ and $A_{circumscribed}$ are estimated values of lower and upper.

(1 mark)

Question 3 **(9 marks)**

Given the following marginal analysis data

$$C'(x) = 3\cos x \sin^2 x + 5e^{3x}, \qquad C(0) = 0$$
$$R'(x) = 6\cos x \sin^2 x + e^{3x} + 2x(10x^2 - 3)^3 + \frac{1}{\cos^2 x}, \qquad R(0) = 0$$

Find $C(x)$ and $R(x)$, with full working out. (6 marks)

Solution

$\int C'(x)dx = C(x)$ (1 mark)

$\rightarrow \int 3\cos x \sin^2 x + 5e^{3x}\ dx$

$\rightarrow 3\int \cos x \sin^2 x\ dx + \frac{5}{3}\int 3e^{3x}\, dx$ (1 mark)

$\rightarrow C(x) = \sin^3 x + \frac{5}{3}e^{3x} + c \qquad C(0) = 0$

$\rightarrow C(0) = \frac{5}{3} + c \rightarrow c = -\frac{5}{3}$

$\therefore C(x) = \sin^3 x + \frac{5}{3}e^{3x} - \frac{5}{3}$ (1 mark)

$\int R'(x)dx = R(x)$ (1 mark)

$\rightarrow \int 6\cos x \sin^2 x + e^{3x} + 2x\left(10x^2 - 3\right)^3 + \frac{1}{\cos^2 x}\ dx$

$\rightarrow 6\int \cos x \sin^2 x\, dx + \frac{1}{3}\int 3e^{3x}\, dx + \frac{1}{10}\int 20x\left(10x^2 - 3\right)^3 dx + \tan x + c$ (1 mark)

$\rightarrow R(x) = 2\sin^3 x + \frac{1}{3}e^{3x} + \frac{1}{40}(10x^2 - 3)^4 + \tan x + c \qquad R(0) = 0$

$\rightarrow R(0) = \frac{1}{3} + \frac{81}{40} + c \rightarrow c = -\frac{283}{120}$

$\therefore R(x) = 2\sin^3 x + \frac{1}{3}e^{3x} + \frac{1}{40}(10x^2 - 3)^4 + \tan x - \frac{283}{120}$ (1 mark)

Hence, $C(x) = \sin^3 x + \frac{5}{3}e^{3x} - \frac{5}{3}$ and $R(x) = 2\sin^3 x + \frac{1}{3}e^{3x} + \frac{1}{40}(10x^2 - 3)^4 + \tan x - \frac{283}{120}$

By finding the cost and revenue function..

a) Find the total cost of producing 10 items (1 mark)

Solution

$C(x) = \sin^3 x + \frac{5}{3}e^{3x} - \frac{5}{3}$

$\therefore C(10) = \$1.78 \times 10^5$ (1 mark)

b) Find the average profit when 10 items are produced and sold. (3 marks)

Solution

$$P(x) = R(x) - C(x) \quad \text{(1 mark)}$$

$$\to P(x) = \left(2\sin^3 x + \frac{1}{3}e^{3x} + \frac{1}{40}(10x^2 - 3)^4 + \tan x - \frac{283}{120}\right) - \left(\sin^3 x + \frac{5}{3}e^{3x} - \frac{5}{3}\right)$$

$$\to P(x) = \sin^3 x - \frac{4}{3}e^{3x} + \frac{1}{40}(10x^2 - 3)^4 + \tan x - \frac{83}{120} \quad \text{(1 mark)}$$

Average profit

$$P_{avg} = \frac{P(10)}{10}$$

$$\therefore P_{avg} = \$ - 1.422 \times 10^{13} \quad \text{(1 mark)}$$

Question 4 **(8 marks)**

The fundamental theorem of calculus is derived by a long algebraic method. To simply explain this, they say that $A = \lim_{n\to\infty}(sum\ of\ areas\ of\ rectangular\ strips)$, depending on the function. This means the exact area A of the region under the curve, which then simplifies to…

$$A = \lim_{\delta x \to 0} \sum_{x=0}^{x=n} f(x)\,\delta x$$

By interpreting on the information given, what does the expression above actually mean. Express your answer in a basic expression. (3 marks)

Solution

If there *are n amount of rectangles created under a curve*, and if there *width δx, is approaching to 0 ($\delta x \to 0$)*, then *the sum of all of them is the estimated area under the curve between $x = 0$ and $x = n$*.

(1 mark)

Hence, if it's the area under a curve then we can say that

$$A = \lim_{\delta x \to 0} \sum_{x=0}^{x=n} f(x)\,\delta x \approx \int_0^n f(x)\,dx$$

(1 mark)

$$\therefore A = \int_0^n f(x)\,dx$$

Hence, the expression above means the area under a curve, which is the integral with limits from $x = 0$ and $x = n$

(1 mark)

By this formula of A, they have deducted the formula for the fundamental theorem of calclus. which is…

$$\frac{d}{dx}(\int_a^x f(t)\,dt) = f(x)$$

Hence, find $f(x)$. (2 marks)

a) $\frac{d}{dx}(\int_{\pi}^{x} \frac{\sqrt[2]{2t^2-4t+3}}{9t-3}\,dt)$

Solution

By using the fundamental theorem of calculus

$$\frac{d}{dx}(\int_a^x f(t)\,dt) = f(x)$$

$$\frac{d}{dx}\left(\int_{\pi}^{x} \frac{\sqrt[2]{2t^2-4t+3}}{9t-3}\,dt\right) = \frac{\sqrt[2]{2x^2-4x+3}}{9x-3}$$

$$\therefore f(x) = \frac{\sqrt[2]{2x^2-4x+3}}{9x-3}$$

(1 mark)

b) $\frac{d}{dx}(\int_{3!}^{x} \frac{1}{2}\left(\frac{\tan(t)+e^{\frac{1}{2}t}-5\left(3t^2-2t\right)^9-10}{\sqrt{4t^2+\sin(t)+10}}\right)dt$

Solution

By using the fundamental theorem of calculus

$$\frac{d}{dx}(\int_a^x f(t)\,dt) = f(x)$$

$$\frac{d}{dx}(\int_{3!}^{x} \frac{1}{2}\left(\frac{\tan(t)+e^{\frac{1}{2}t}-5(3t^2-2t)^9-10}{\sqrt{4t^2+\sin(t)+10}}\right)dt$$

$$= \frac{1}{2}\left(\frac{\tan(x)+e^{\frac{1}{2}x}-5(3x^2-2x)^9-10}{\sqrt{4x^2+\sin(x)+10}}\right)$$

$$\therefore f(x) = \frac{\tan(t)+e^{\frac{1}{2}t}-5\left(3t^2-2t\right)^9-10}{2\sqrt{4t^2+\sin(t)+10}}$$

(1 mark)

Why doesn't the expressions below not work for the fundamental theorem of calculus?

$$\frac{d}{dx}(\int_3^{x^2}\left(\frac{t}{t+1}\right)dt \qquad and \qquad \frac{d}{dx}(\int_1^{x}\left(\frac{\sqrt{2t-4}}{t+1}\right)dt$$

Hence, evaluate the real expression from the two. (2 marks)

Solution

$\frac{d}{dx}(\int_1^{x}\left(\frac{\sqrt{2t-4}}{t+1}\right)dt$ this expression doesn't work, as it's undefined (Evaluated by using CAS)

(1 mark)

(Proof, no need to show this)

$$\int_1^{x}\left(\frac{\sqrt{2t-4}}{t+1}\right)dt = \left[-2\sqrt{6}\,tan^{-1}\left(\frac{\sqrt{3t-6}}{3}\right)+2\sqrt{2t-4}\right]_1^x$$

$$\rightarrow\left(-2\sqrt{6}\,tan^{-1}\left(\frac{\sqrt{3(x)-6}}{3}\right)+2\sqrt{2(x)-4}\right)-\left(-2\sqrt{6}\,tan^{-1}\left(\frac{\sqrt{3(1)-6}}{3}\right)+2\sqrt{2(1)-4}\right)$$

Hence, by solving the integral by the lower limit specifically, it shows the integral is undefined.

$$\frac{d}{dx}(\int_3^{x^2}\left(\frac{t}{t+1}\right)dt = 2x\left(\frac{x}{x+1}\right) = \frac{2x^2}{x+1}$$

(Can be done by using CAS)

$$\therefore f(x) = \frac{2x^2}{x+1}$$

(1 mark)

END OF CALCULATOR-ASSUMED

Additional working space

Question number: __________